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Solutions of the non-additive YBE by spectral parameter factorization and re-Yang–Baxterization

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Abstract. R-matrix solutions for the non-additive Yang–Baxter equation are obtained in arbitrary dimension using a factorization ansatz in spectral parameters. A symmetry transformation determined for the 'particle conserving' R-matrix is used to formulate a re-Yang–Baxterization scheme with multicomponent spectral parameters, extending the notion of conventional Yang–Baxterization. The possible scope for such a transformation, which has a striking resemblance to Reshetikhin's deformation of R-matrices but extended now to include spectral parameters, is discussed.

1. Introduction

The importance of the Yang–Baxter equation (YBE), after its successful application to statistical and quantum integrable systems, see for example [1], has been reaffirmed recently due to the major role it has played in diverse subjects like quantum groups, braid groups, link and knot polynomials etc, see for example [2]. In general the Yang–Baxter equation may be given through the elements of the quantum R-matrix as

$$\sum_{k_1,k_2,k_3} R_{i_1 i_2}^{k_1 k_2}(\lambda,\mu) R_{k_1 i_3}^{j_1 k_3}(\lambda,\gamma) R_{k_2 k_3}^{j_2 j_3}(\mu,\gamma) = \sum_{l_1,l_2,l_3} R_{l_1 l_2}^{j_1 j_2}(\lambda,\mu) R_{i_1 l_3}^{l_1 j_3}(\lambda,\gamma) R_{i_2 i_3}^{l_2 l_3}(\mu,\gamma) \tag{1.1}$$

where the indices run from 1 to N and λ, μ, γ are in general multicomponent spectral parameters. However, most of the solutions of the YBE considered in the literature are of additive type, i.e. $R(\lambda,\mu) \equiv R(\lambda-\mu)$ with single-component spectral parameters. Nevertheless some important classes of integrable models related to both elliptic and higher genus curves have also been investigated, in which non-additive solutions with multicomponent spectral parameters were obtained [3].

On the other hand it has been recently discovered, that starting from the spectral-parameterless braid group solution, and using the Yang–Baxterization procedure, it is possible to construct an additive type $(N^2 \times N^2)$ R-matrix expressed in trigonometric functions [4]. Though the parallel procedure for the non-additive case is not yet so well formulated, some interesting investigations leading to explicit YBE solutions of this type have been carried out for $N = 2$ [5, 6]. Therefore it is natural to seek some generalization of such non-additive $R(\lambda,\mu)$-matrix solutions for arbitrary N. At the

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same time it would also be encouraging to explore the possible Yang–Baxterization procedure for systematically generating such solutions, starting from the braid group representations, and generalizing them to the multicomponent spectral parameter case.

These objectives motivated us to use an interesting spectral parameter factorization ansatz of R-matrix elements to find an explicit solution of the non-additive YBE (1.1) for arbitrary N. It is worth mentioning that at the limit $\lambda = \mu$ such solutions yield multiparameter deformations of the standard [7] as well as the graded or 'exotic' braid group solutions [6, 8] related to gl(N). By restricting our investigation to the 'particle conserving' case we are able to find some spectral parameter dependent symmetry transformations of the R-matrix, which in turn helps to formulate a re-Yang–Baxterization scheme for producing more general solutions of the YBE. We discuss also the similarity of such transformations with Reshetikhin's construction of the deformed R-matrix [9] and its possible implications.

The arrangement of this paper is as follows. In section 2 the proposed factorization ansatz is used to solve the non-additive YBE for the case of higher N. Section 3 finds the explicit R-matrix solution and the corresponding braid group representation. In section 4 the symmetry transformation is explored and the re-Yang–Baxterization scheme is presented. Section 5 gives the concluding remarks.

2. The factorization ansatz and its application to the YBE

Since the YBE actually represents a collection of a large number of equations, it is, in general, difficult to find explicit solutions of it without assuming some symmetry condition. As we aim to find solutions of equation (1.1), which would contain the braid group realization of gl(N) in the standard representation, we may restrict ourselves to both the 'particle conserving' and the triangular form of the R-matrix. In element notation this means that $R_{ij}^{kl} \neq 0$ only when the incoming indices (i, j) are same as the outgoing ones (k, l), modulo their permutations, and moreover, $R_{ij}^{ji} \neq 0$, when $i < j$ $(i > j)$ for the upper (lower) triangular case. Since now in the YBE the incoming indices (i_1, i_2, i_3) must be some permutations of the outgoing ones (j_1, j_2, j_3) the resulting equations are considerably simplified and may be grouped together in the following way.

Evidently the set of equations where all the three incoming 'particles' or indices are identical are trivially satisfied. However those with only two coinciding indices (say i) and the third being different (say j) give the non-trivial equations

$$R_{ii}^{ii} R'^{ij}_{ij} = R_{ij}^{ij} R'^{ii}_{ii} \tag{2.1a}$$

$$R'^{ii}_{ii} R''^{ji}_{ji} = R'^{ji}_{ji} R''^{ii}_{ii} \tag{2.1b}$$

$$R_{ii}^{ii} R'^{ji}_{ij} R''^{ii}_{ii} = R_{ji}^{ji} R'^{ji}_{ij} R''^{ij}_{ij} + R_{ij}^{ji} R'^{ii}_{ii} R''^{ji}_{ij} \tag{2.1c}$$

$$R_{ii}^{ii} R'^{ij}_{ji} R''^{ii}_{ii} = R_{ji}^{ji} R'^{ij}_{ji} R''^{ij}_{ij} + R_{ji}^{ij} R'^{ii}_{ii} R''^{ij}_{ji} \tag{2.1d}$$

where we have introduced the shorthand notation $R = R(\lambda, \mu)$, $R' = R(\lambda, \gamma)$, $R'' = R(\mu, \gamma)$ for the arguments of the R-matrix elements and we have chosen in particular the upper triangular case. Note that for $N = 2$, when the indices can take only two different values, the equations (2.1*a*)–(2.1*d*) constitute the complete set. On the other hand, for the higher N case another situation where all the three incoming indices are different ($i \neq j \neq k$) should also be considered along with this set. Such an additional set takes the form

$$\begin{aligned}
&R_{ij}^{ij} R'^{ik}_{ik} = R_{ik}^{ik} R'^{ij}_{ij} \qquad R'^{jk}_{jk} R''^{ik}_{ik} = R'^{ik}_{ik} R''^{jk}_{jk} \\
&R_{ij}^{ji} R'^{jk}_{jk} R''^{ki}_{ik} = R_{jk}^{jk} R'^{ji}_{ij} R''^{kj}_{jk} + R_{kj}^{jk} R'^{ki}_{ik} R''^{jk}_{jk} \\
&R_{ik}^{ki} R'^{ij}_{ij} R''^{kj}_{jk} = R_{ij}^{ij} R'^{ki}_{ik} R''^{ij}_{ji} + R_{ij}^{ji} R'^{kj}_{jk} R''^{ij}_{ij} \\
&R_{ij}^{ij} R'^{ki}_{ik} R''^{ji}_{ji} + R_{ij}^{ji} R'^{kj}_{jk} R''^{ji}_{ij} = R_{kj}^{kj} R'^{ki}_{ik} R''^{jk}_{jk} + R_{jk}^{kj} R'^{ji}_{ij} R''^{kj}_{jk}.
\end{aligned} \tag{2.2}$$

We now propose an ansatz for finding explicit solutions for the R-matrix, where each element of it is factorized in spectral parameters as $R_{ij}^{kl}(\lambda,\mu) = f_{ij}^{kl}(\lambda) g_{ij}^{kl}(\mu)$. Curiously this ansatz leads to the separation of spectral parameters allowing decoupling of three independent variables in the YBE and highly simplifies these algebraic equations, just in analogy with the separation of variables in partial differential equations. For example equations (2.1*a*)–(2.1*b*) now give

$$\frac{f_{ii}^{ii}(\lambda)}{f_{ji}^{ji}(\lambda)} = \frac{f_{ii}^{ii}(\mu)}{f_{ji}^{ji}(\mu)} \qquad \frac{g_{ii}^{ii}(\lambda)}{g_{ij}^{ij}(\lambda)} = \frac{g_{ii}^{ii}(\mu)}{g_{ij}^{ij}(\mu)} \tag{2.3}$$

with the interesting consequence

$$f_{ii}^{ii}(\lambda) = c_{ij} f_{ji}^{ji}(\lambda) \qquad g_{ii}^{ii}(\lambda) = d_{ij} g_{ij}^{ij}(\lambda) \tag{2.4}$$

where c_{ij} and d_{ij} are constants independent of spectral parameters. Application of the factorization ansatz along with the relation (2.4) reduces in turn equations (2.1*c*)–(2.1*d*) to the form

$$\begin{aligned}
f_{ij}^{ji}(\lambda) g_{ij}^{ji}(\lambda) &= \{ f_{ii}^{ii}(\lambda) g_{ii}^{ii}(\lambda) - \Delta_{ij}^{-1} f_{jj}^{jj}(\lambda) g_{jj}^{jj}(\lambda) \} \\
&= \{ f_{jj}^{jj}(\lambda) g_{jj}^{jj}(\lambda) - \Delta_{ij}^{-1} f_{ii}^{ii}(\lambda) g_{ii}^{ii}(\lambda) \}
\end{aligned} \tag{2.5}$$

where $\Delta_{ij} = c_{ij} c_{ji} d_{ij} d_{ji}$.

As already mentioned, for higher N values one should also consider the set of equations (2.2), where we may again apply our factorization ansatz together with the above reduced relations (2.4) and (2.5). As a result one finally obtains

$$\frac{f_{kj}^{jk}(\lambda) f_{ji}^{ij}(\lambda)}{f_{jj}^{jj}(\lambda) f_{ki}^{ik}(\lambda)} = \frac{f_{kj}^{jk}(\mu) g_{ki}^{ik}(\mu)}{f_{jj}^{jj}(\mu) g_{ji}^{ij}(\mu)} \tag{2.6a}$$

$$\frac{g_{kj}^{jk}(\lambda) g_{ji}^{ij}(\lambda)}{g_{jj}^{jj}(\lambda) g_{ki}^{ik}(\lambda)} = \frac{g_{kj}^{jk}(\mu) f_{ki}^{ik}(\mu)}{g_{jj}^{jj}(\mu) f_{ji}^{ij}(\mu)} \tag{2.6b}$$

$$\Delta_{ik} f_{jj}^{jj}(\lambda) g_{jj}^{jj}(\lambda) = \Delta_{kj} f_{ii}^{ii}(\lambda) g_{ii}^{ii}(\lambda) = \Delta_{ij} f_{kk}^{kk}(\lambda) g_{kk}^{kk}(\lambda). \tag{2.6c}$$

It may be noted that while (2.4) and (2.6*c*) establish relations between the diagonal elements of f and g, (2.5) and (2.6*a*), (2.6*b*) relate diagonal entries to the non-diagonal ones. In the next section we will explore explicit solutions of the YBE for the general N case using the relations derived here.

3. Explicit solutions of the non-additive YBE for higher N values

It is evident from (2.6*c*) that the products of functions $f_{ii}^{ii}(\lambda)g_{ii}^{ii}(\lambda)$ for any value of i are related to each other and may be expressed as

$$f_{ii}^{ii}(\lambda)g_{ii}^{ii}(\lambda) = \rho_i \kappa(\lambda) \tag{3.1}$$

where the ρ_i's are λ-independent constants. This form in turn reduces (2.6*c*) itself to $\Delta_{ik}\rho_j = \Delta_{kj}\rho_i = \Delta_{ij}\rho_k$ suggesting a possible solution $\Delta_{ij} = \rho_i\rho_j$. Using these results we can rewrite (2.5) as

$$f_{ij}^{ji}(\lambda)g_{ij}^{ji}(\lambda) = \left(\rho_i - \frac{1}{\rho_i}\right)\kappa(\lambda) = \left(\rho_j - \frac{1}{\rho_j}\right)\kappa(\lambda) \tag{3.2}$$

which interestingly restricts the values of ρ_i through a quadratic equation $\rho_i - \rho_i^{-1} =$ constant. Clearly the two roots $\rho_i^{\pm}$ allowed are related by $\rho_i^{+} = -(\rho_i^{-})^{-1} \equiv q$. We will see later an important consequence of these two different solutions; namely the case when all the ρ_i's belong to the same root corresponds to the standard braid group representation of $gl(n)$, while the graded or 'exotic' solutions emerge when different root sectors are involved with at least one $\Delta_{ij} = \rho_i\rho_j = -1$. We may look again at (3.2), which has now reduced to a single equation $f_{ij}^{ji}(\lambda)g_{ij}^{ji}(\lambda) = (q - q^{-1})\kappa(\lambda)$ suggesting for the individual elements a form

$$f_{ij}^{ji}(\lambda) = (q - q^{-1})\kappa(\lambda)u_{ij}(\lambda) \qquad g_{ij}^{ji}(\lambda) = \frac{1}{u_{ij}(\lambda)} \tag{3.3}$$

where $u_{ij}(\lambda)$ are for the time being arbitrary functions. We now use equations (2.6*a*) and (2.6*b*) which relate the diagonal elements to the non-diagonal ones and give

$$f_{jj}^{jj}(\lambda) = c_j(q - q^{-1})\kappa(\lambda)\frac{u_{kj}(\lambda)u_{ji}(\lambda)}{u_{ki}(\lambda)} \qquad g_{jj}^{jj}(\lambda) = d_j\frac{u_{ki}(\lambda)}{u_{kj}(\lambda)u_{ji}(\lambda)} \tag{3.4}$$

c_j, d_j being some constants. A closer look at the above relations (3.4) shows that the LHSs are independent of the indices k and i and therefore a natural choice for u_{ij} is the factorized form $u_{ij}(\lambda) = u_i^{(1)}(\lambda)u_j^{(2)}(\lambda)$, where $u_i^{(a)}$, $a = 1, 2$ are $2N$ independent functions. Such a choice directly results in

$$f_{jj}^{jj}(\lambda) = c_j(q - q^{-1})\kappa(\lambda)u_j^{(1)}(\lambda)u_j^{(2)}(\lambda) \qquad g_{jj}^{jj}(\lambda) = d_j\frac{1}{u_j^{(1)}(\lambda)u_j^{(2)}(\lambda)} \tag{3.5}$$

where, due to equation (3.1), the constants become related by $(q - q^{-1})c_jd_j = \rho_j$. In a similar way we may easily obtain the solutions for the elements g_{ij}^{ij} and f_{ij}^{ij} through the functions $u_i^{(a)}$ using (2.4). Collecting all the explicit solutions above for different elements of f and g we finally arrive at a non-additive solution of the YBE, which can be expressed in matrix form as

$$\begin{aligned} R(\lambda,\mu) &= \sum R_{ij}^{kl}(\lambda,\mu)e_{ik}\otimes e_{jl} \\ &= \sum_i \rho_i \frac{u_i^{(1)}(\lambda)u_i^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_i^{(2)}(\mu)}e_{ii}\otimes e_{ii} + \sum_{i\neq j}\phi_{ij}\frac{u_j^{(1)}(\lambda)u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_i^{(2)}(\mu)}e_{ii}\otimes e_{jj} \\ &\quad + (q - q^{-1})\sum_{i<j}\frac{u_i^{(1)}(\lambda)u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_j^{(2)}(\mu)}e_{ij}\otimes e_{ji} \end{aligned} \tag{3.6}$$

where $\phi_{ij} = (q - q^{-1})c_j d_i / c_{ji} d_{ij}$, e_{ij} form the basis of gl(N) with $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and the solution is normalized by a factor $\kappa(\lambda)$. Some important comments regarding this R-matrix solution of the YBE (1.1) for arbitrary values of N are in order. Firstly, the factorized form of λ, μ dependence is explicit in (3.6). Secondly, the structure of constant parameters ϕ_{ij} clearly induces a restriction $\phi_{ij}\phi_{ji} = 1$ resulting in the inclusion of $\frac{1}{2}N(N-1)$ free parameters in the solution. We should note that the constant multiplicative deformation of the 'particle conserving' R-matrix was also considered earlier [10] and an explicit solution with deforming parameters like ϕ_{ij} was first found in [11] for the additive case, as a generalization of the six-vertex model. However one may observe that in our non-additive solution (3.6) $2N$ additional arbitrary functions $u_i^{(a)}(\lambda)$ are also present, which may be considered as the spectral parameter dependent extension of the constant deformations [11]. It is also relevant to note that at $\lambda = \mu$ the above solution directly yields the braid group realization related to gl(N) in the standard representation

$$R = \sum_i \rho_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} \phi_{ij} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}. \tag{3.7}$$

We recall that the ρ_i's can only be taken either as q or $-q^{-1}$. Interestingly if we choose $\rho_i = q$ for $i \in [1, \ldots, m]$ and $\rho_j = -q^{-1}$ for $j \in [m+1, \ldots, m+n]$ with $m + n = N$, resulting in $\Delta_{ij} = \rho_i \rho_j = -1$, one gets from (3.7) a braid group solution related to graded gl$(m \mid n)$ [8], but with multiparameter deformation. However if all the ρ_i's are chosen to be q or $-q^{-1}$, we recover the standard braid group realization of gl(N) [7]. Note that similar deformations was obtained earlier through the twisted quasitriangular Hopf algebra [7, 9] and the graded solutions for the additive case was found in [11].

It is natural to expect that the known non-additive solutions in the two dimensional case [5, 6] will be recovered from our general solution (3.6) for $N = 2$. We observe that in the standard situation, i.e. when $\rho_1 = \rho_2 = q$ with $\Delta_{12} = q^2$, the earlier result is indeed obtainable as a reduction. However in the 'exotic' sector, i.e. when $\rho_1 = q$, $\rho_2 = -q^{-1}$ with $\Delta_{12} = -1$ the solution we get as a reduction of (3.6) does not represent the general form obtained earlier [5, 6] for this case. To understand why this happens, we notice first that for $N = 2$ the 'three particle' equations (2.2) become redundant, lifting certain restrictions on the form of the solution. We also find that (2.5) gives

$$(1 + \Delta_{12}^{-1})(f_{11}^{11}(\lambda) g_{11}^{11}(\lambda) - f_{22}^{22}(\lambda) g_{22}^{22}(\lambda)) = 0 \tag{3.8}$$

and at $\Delta_{12} = -1$ the relation between the diagonal elements, similar to (3.1), is not essential. Therefore in this sector relations like (3.3) may now be modified to

$$f_{12}^{21}(\lambda) = f_{11}^{11}(\lambda) u(\lambda) \qquad g_{12}^{21}(\lambda) = g_{11}^{11}(\lambda) \left(1 + \frac{g_{22}^{22}(\lambda) f_{22}^{22}(\lambda)}{g_{11}^{11}(\lambda) f_{11}^{11}(\lambda)}\right) \frac{1}{u(\lambda)} \tag{3.9}$$

resulting finally the solution for the R-matrix in the form

$$\begin{aligned} R(\lambda, \mu) = {} & e_{11} \otimes e_{11} + f(\lambda) g(\mu) e_{22} \otimes e_{22} \\ & + q f(\lambda) e_{11} \otimes e_{22} - q^{-1} g(\mu) e_{22} \otimes e_{11} + \{1 + f(\mu) g(\mu)\} \frac{u(\lambda)}{u(\mu)} e_{12} \otimes e_{21} \end{aligned} \tag{3.10}$$

where f, g and u are arbitrary independent functions with the notation $f(\lambda) = f_{22}^{22}(\lambda)/f_{11}^{11}(\lambda)$, $g(\lambda) = g_{22}^{22}(\lambda)/g_{11}^{11}(\lambda)$. Note that the above solution finally recovers the earlier result and though it gives the same 'exotic' braid group representation as before, it is more general than the $N = 2$ reduction of (3.6), reflecting a special freedom in the two-dimensional case. After obtaining the explicit solution for arbitrary N in the triangular form, we intend to study next the symmetries of the YBE for generating more general 'particle conserving' R-matrices with multicomponent spectral parameters.

4. Re-Yang–Baxterization through symmetry transformation

Given a braid group solution it is a natural to pose the problem of immediately generating a non-additive type spectral parameter dependent solution of the YBE, or in other words Yang–Baxterize it using some symmetry relation of the equation. We recall that a similar problem has been studied extensively in the literature for the additive case [4] and attempt to present here an analogous scheme for the non-additive case, which not only would justify the existence of solutions like (3.6) but also represent some re-Yang–Baxterization procedure capable of yielding a class of R-matrices with multicomponent spectral parameters.

Motivated by the factorized form of the solution (3.6), we consider the following transformation of the R-matrix

$$\tilde{R}_{ij}^{kl}(\lambda_1, \lambda_2; \mu_1, \mu_2) = R_{ij}^{kl}(\lambda_1, \mu_1) \frac{u_l^{(1)}(\lambda_2) u_j^{(2)}(\lambda_2)}{u_i^{(1)}(\mu_2) u_k^{(2)}(\mu_2)} \tag{4.1}$$

with $2N$ arbitrary functions $u_i^{(a)}(\lambda)$ and ask whether the bi-component spectral parameter dependent $\tilde{R}$-matrix would be a solution of the YBE (1.1), if the original $R(\lambda_1, \mu_1)$ is an exact solution of it. Inserting (4.1) directly into the equation one finds that it is possible to put the following sufficient conditions on separate spectral parameter dependent entries

$$u_{i_2}^{(2)}(\lambda_2) u_{i_3}^{(2)}(\lambda_2) u_{k_2}^{(1)}(\lambda_2) u_{k_3}^{(1)}(\lambda_2) = u_{j_2}^{(1)}(\lambda_2) u_{j_3}^{(1)}(\lambda_2) u_{l_2}^{(2)}(\lambda_2) u_{l_3}^{(2)}(\lambda_2) \tag{4.2a}$$

$$u_{k_1}^{(1)}(\gamma_2) u_{k_2}^{(1)}(\gamma_2) u_{j_1}^{(2)}(\gamma_2) u_{j_2}^{(2)}(\gamma_2) = u_{l_1}^{(2)}(\gamma_2) u_{l_2}^{(2)}(\gamma_2) u_{i_1}^{(1)}(\gamma_2) u_{i_2}^{(1)}(\gamma_2) \tag{4.2b}$$

$$u_{j_3}^{(1)}(\mu_2) u_{k_3}^{(2)}(\mu_2) / u_{i_1}^{(1)}(\mu_2) u_{k_1}^{(2)}(\mu_2) = u_{i_3}^{(2)}(\mu_2) u_{l_3}^{(1)}(\mu_2) / u_{j_1}^{(2)}(\mu_2) u_{l_1}^{(1)}(\mu_2) \tag{4.2c}$$

where (i_1, i_2, i_3) are incoming and (j_1, j_2, j_3) outgoing indices with the rest being intermediate ones. To satisfy these equations we assume the 'particle conserving' condition on the initial R-matrix without necessarily limiting it to triangular form. This assumption, as evident from (1.1), obviously restricts the possible values of intermediate indices like k_i and l_i yielding relations like

$$u_{k_2}^{(1)}(\lambda_2) u_{k_3}^{(1)}(\lambda_2) = u_{j_2}^{(1)}(\lambda_2) u_{j_3}^{(1)}(\lambda_2) \qquad u_{l_2}^{(2)}(\lambda_2) u_{l_3}^{(2)}(\lambda_2) = u_{i_2}^{(2)}(\lambda_2) u_{i_3}^{(2)}(\lambda_2)$$

due to which the condition (4.2a) is clearly satisfied. In an exactly similar way the remaining relations (4.2b) and (4.2c) can also be shown to hold, consequently proving that $\bar{R}(\lambda, \mu)$ is a genuine solution of the YBE for the 'particle conserving' case. At the cost of restriction on the deforming functions $u_i^{(a)}(\lambda)$, more general assumptions may possibly be posed; however these will not be discussed here. One may note that there exists a spectral parameter dependent similarity transformation of the form

$$R(\lambda, \mu) \to [S(\lambda_p) \otimes S(\mu_p)] R(\lambda, \mu) [S(\lambda_p) \otimes S(\mu_p)]^{-1}$$

which keeps the solution of the non-additive YBE (1.1) invariant. Applying such a transformation on the $\bar{R}(\lambda, \mu)$-matrix given in (4.1) and choosing $S_{ij}(\lambda_2) = \delta_{ij} u_i^{(2)}(\lambda_2)$, the solution (4.1) may also be written in a simplified form as

$$\bar{R}_{ij}^{kl}(\lambda_1, \lambda_2; \mu_1, \mu_2) = R_{ij}^{kl}(\lambda_1, \mu_1) \frac{u_l(\lambda_2)}{u_i(\mu_2)}$$

where $u_l(\lambda) = u_l^{(1)}(\lambda) u_l^{(2)}(\lambda)$ are now N arbitrary functions, one of which may also be removed through an overall scaling [12].

We return again to the form (4.1) and consider some of its relevant consequences when the components of spectral parameters are coincident. This gives

$$\bar{R}_{ij}^{kl}(\lambda, \mu) = R_{ij}^{kl}(\lambda, \mu) \frac{u_l^{(1)}(\lambda) u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu) u_k^{(2)}(\mu)} \tag{4.3}$$

which now represents a transformation between single component R-matrices, reflecting the symmetry of the non-additive YBE in the 'particle conserving' case. It may be noticed that if one imposes 'charge conjugation'-, 'parity'- and 'time reversal'-like symmetries on the original R-matrix, they are evidently broken by the transformation (4.3) and therefore it may be considered as some non-additive type of generalization of the 'symmetry breaking transformation' [13]. To see this in more detail, let us write (4.3) explicitly for $N = 2$ by taking the original R-matrix as the known trigonometric six-vertex solution

$$\begin{aligned}
&\bar{R}_{11}^{11} = \sin(\lambda - \mu + \alpha) \frac{D(\lambda)}{D(\mu)} \qquad &&\bar{R}_{22}^{22} = \sin(\lambda - \mu + \alpha) \frac{D(\mu)}{D(\lambda)} \\
&\bar{R}_{12}^{12} = \sin(\lambda - \mu) \frac{1}{D(\lambda) D(\mu)} \qquad &&\bar{R}_{21}^{21} = \sin(\lambda - \mu) D(\lambda) D(\mu) \\
&\bar{R}_{12}^{21} = \bar{R}_{21}^{12} = \sin\alpha
\end{aligned} \tag{4.4}$$

where we have introduced $D(\lambda) = u_1(\lambda)/u_2(\lambda)$ assuming a reduction $u_i^2 = u_i^1 \equiv u_i$ and used the normalization freedom of the R-matrix. Further restricting $D(\lambda)$ to be independent of the spectral parameter, we easily find that only the elements $\bar{R}_{12}^{12}$ and $\bar{R}_{21}^{21}$ are now deformed as the 'β-transformation' of [13] and give the deformed six-vertex solution [11]. On the other hand if we assume $u_i^{(2)}(\lambda) = \left(u_i^{(1)}(\lambda)\right)^{-1}$ in (4.3) the only terms which would suffer deformation are $\bar{R}_{12}^{21}(\lambda, \mu) = \sin\alpha F(\mu)/F(\lambda)$ and $\bar{R}_{21}^{12}(\lambda, \mu) = \sin\alpha F(\lambda)/F(\mu)$. The requirement of additivity of spectral parameters in the R-matrix naturally gives the choice $F(\lambda) = e^{i\lambda\theta}$ reproducing the 'α-transformation' [13]. Another interesting fact about solution (4.4) is that it recovers exactly the known non-additive solution, important in the context of physical systems like ferroelectrics in an electric field, the two dimensional Ising model with nearest and next-nearest neighbour interactions etc [14]. Therefore, the transformation (4.3) found here, being valid for arbitrary N, may be expected to have physical consequences in a more general situation.

Returning to equation (4.1) it may also be observed that by the repeated use of this formula it is possible to formulate a re-Yang–Baxterization scheme for generating multicomponent spectral parameter dependent R-matrices starting from a braid group representation given by

$$\tilde{R}_{ij}^{kl}(\lambda_1,\lambda_2,\ldots,\lambda_s;\mu_1,\mu_2,\ldots,\mu_s) = R_{ij}^{kl}\prod_{p=1}^{s}\frac{u_l^{(1,p)}(\lambda_p)u_j^{(2,p)}(\lambda_p)}{u_i^{(1,p)}(\mu_p)u_k^{(2,p)}(\mu_p)}. \tag{4.5}$$

The appearance of functions like $u_i^{(a,p)}(\lambda_p)$ shows that at each step of the iteration one may introduce new arbitrary functions of additional spectral parameters. If however we consider only $s = 1$, a more conventional Yang–Baxterization scheme is obtained with a single component spectral parameter, though generalized now for the non-additive case. We also see that such a Yang–Baxterization can recover directly the explicit solution obtained in (3.6) starting from the braid group representation of gl(N). On the other hand, we may also start from known additive R-matrices already Yang–Baxterized by the standard method [4] and re-Yang–Baxterize it using (4.5). For example, considering the R-matrix of the six vertex model and iterating it once we may obtain the bi-component spectral parameter solution given in [5].

Before focusing on the potential implications of our multi-spectral parameter deformation (4.5), let us recall briefly the main points of Reshitikhin's multiparameter deformation of the R-matrix related to the twisted Hopf algebra [9] in simpler matrix language [7]. Given a braid group representation R, one can construct a new representation as

$$\tilde{R} = F^{-1}RF^{-1} \tag{4.6}$$

if the deforming matrix F itself satisfies the braid group like equations

$$F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12} \qquad R_{12}F_{13}F_{23} = F_{23}F_{13}R_{12} \qquad F_{12}F_{13}R_{23} = R_{23}F_{13}F_{12} \tag{4.7}$$

with the constraint $F_{12}F_{21} = 1$. Returning now to our relation (4.5) and for simplicity restricting ourselves to only the first iteration, we may write it in the following matrix form

$$R(\lambda,\mu) = F^{-1}(\lambda,\mu)R\tilde{F}^{-1}(\lambda,\mu) \tag{4.8}$$

taking R as the braid group solution (3.7) and the deforming matrices F, $\tilde{F}$ expressed as

$$F(\lambda,\mu) = \sum_{ij}\frac{u_i^{(1)}(\mu)}{u_j^{(2)}(\lambda)}e_{ii}\otimes e_{jj} \qquad \tilde{F}(\lambda,\mu) = \sum_{ij}\frac{u_i^{(2)}(\mu)}{u_j^{(1)}(\lambda)}e_{ii}\otimes e_{jj}.$$

Clearly the requirement $u_i^{(1)} = u_i^{(2)} \equiv u_i$ makes $F(\lambda,\mu) = \tilde{F}(\lambda,\mu)$ giving a striking similarity of (4.8) to Reshetikhin's construction (4.6), but now with the inclusion of spectral parameters. More interestingly, the related $F(\lambda,\mu)$ matrix now satisfies the spectral parameter dependent 'braid group like' equations analogous to (4.7)

$$
\begin{aligned}
&F_{12}(\lambda,\mu)F_{13}(\lambda,\gamma)F_{23}(\mu,\gamma) = F_{23}(\mu,\gamma)F_{13}(\lambda,\gamma)F_{12}(\lambda,\mu)\\
&R_{12}F_{13}(\lambda,\gamma)F_{23}(\mu,\gamma) = F_{23}(\mu,\gamma)F_{13}(\lambda,\gamma)R_{12}\\
&F_{12}(\lambda,\mu)F_{13}(\lambda,\gamma)R_{23} = R_{23}F_{13}(\lambda,\gamma)F_{12}(\lambda,\mu)
\end{aligned}
\tag{4.9}
$$

along with the constraint $F_{12}(\lambda,\mu)F_{21}(\mu,\lambda) = 1$, resembling again the constraint of Reshitikhin. Moreover, even in the general case of (4.5) with arbitrary s, similar reasoning applies resulting in a generalization to the multicomponent spectral parameter. Note however that here the deforming matrix F, contrary to Reshetikhin's construction, may be expressed as the direct product: $F(\lambda,\mu) = A(\mu)\otimes A^{-1}(\lambda)$ with $A(\lambda) = \sum_i u_i(\lambda)e_{ii}$, also showing why the parameters in the standard deformation [7] are greater in number $(\frac{1}{2}N(N-1))$ than the $2N$ number of parameters found here. On the other hand the salient feature of our construction is that the deformations here are spectral parameter dependent in contrast with the known multiparameter deformation [7, 9].

This opens up the promising possibility of extending Reshetikhin's approach to the inclusion of spectral parameters, leading up to the notion of a parametrized twisted Hopf algebra.

5. Concluding remarks

A factorization ansatz in spectral parameters is shown to be effective for finding explicit $(N^2 \times N^2)$ R-matrix solution of the non-additive YBE. Structurally such solutions fall into two classes yielding at the braid group limit either the standard or the graded solutions related to gl(N) with multiparameter deformation [7].

An interesting symmetry transformation is found to exist for the 'particle conserving' R-matrix with arbitrary, spectral parameter dependent, deforming functions. Such functions may be considered as the spectral parameter dependent extension of the constant deforming parameters [11] appearing in the additive case. Repeated use of the symmetry transformation consequently presents a re-Yang–Baxterization scheme extending the notion of conventional Yang–Baxterization—from single to multi-component as well as from additive to non-additive spectral parameters. On the other hand they also have a remarkable analogy with Reshetikhin's construction of the deformed R-matrix related to the twisted quasitriangular Hopf algebra, which might lead to its possible spectral parametrized extension.

It is desirable however to investigate similar non-additive solutions related to braid group solutions other than gl(N), including their higher representations and to explore the possible symmetry transformations. We hope that the non-additive R-matrices studied here might also be applicable to the spectral parametrized approach [15] of Faddeev–Reshitikhin–Takhtajan construction, as well as for constructing Lax operators of quantum integrable models exploiting the underlying quantum group structures [16], as has already been done in the traditional additive case.

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